

Electromagnetism

Problems and solutions

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Morgan & Claypool Publishers

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ISBN 978-1-6817-4429-2 (ebook)

ISBN 978-1-6817-4428-5 (print)

ISBN 978-1-6817-4431-5 (mobi)

DOI 10.1088/978-1-6817-4429-2

Version: 20161101

IOP Concise Physics

ISSN 2053-2571 (online)

ISSN 2054-7307 (print)

A Morgan & Claypool publication as part of IOP Concise Physics

Published by Morgan & Claypool Publishers, 40 Oak Drive, San Rafael, CA, 94903 USA

IOP Publishing, Temple Circus, Temple Way, Bristol BS1 6HG, UK

To my family, my mentors, and my students — CCI

To my friends, family, and mentors — ZSS

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Preface

We wrote this book of problems and solutions having in mind the undergraduate student—sophomore, junior, or senior—who may want to work on more problems and receive immediate feedback while studying. The authors strongly recommend the textbook by David J Griffiths, *Introduction to Electrodynamics*, as a first source manual, since it is recognized as one of the best books on electrodynamics at the undergraduate level. We consider this book of problems and solutions a companion volume for the student who would like to work on more electrostatic problems by herself/himself in order to deepen their understanding and problems solving skills. We add brief theoretical notes and formulae; for a complete theoretical approach we suggest Griffiths' book. Every chapter is organized as follows: brief theoretical notes followed by the problem text with the solution. Each chapter ends with a brief bibliography.

We plan to write a second volume on electrodynamics, which will start with Maxwell's equations and the conservation laws, and then discuss electromagnetic (EM) waves, potentials and fields, radiation, and relativistic electrodynamics.

We follow here the notation of Griffiths, and use \hat{r} for the vector from a source point \vec{r}' to the field point \vec{r} . Please note that $\hat{r} = \frac{\vec{r}}{r} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$ and, as you see, this notation already greatly simplifies complex equations, but you need to be careful with your notation, in particular if you only use cursive or typed letters. Also, we use the same notation s for the distance to the z -axis in cylindrical coordinates as is used in Griffiths' book.

The chosen units are SI units—the international system. The reader should be aware that other books may employ either the Gaussian system (CGS) or the Heaviside–Lorentz (HL) system. The Coulomb force in each of the systems is as follows,

SI system:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

CGS:

$$\vec{F} = \frac{q_1 q_2}{r^2} \hat{r}$$

HL:

$$\vec{F} = \frac{1}{4\pi} \frac{q_1 q_2}{r^2} \hat{r}$$

Some of the problems are typical practice problems with the pedagogical role of improving understanding and problem solving skills. Several of the problems presented here appear in a variety of undergraduate textbooks on EM as they are classic examples; however, we felt it would be incomplete to omit these problems as

they are fundamental to the study of EM. We also present problems that are more general in nature, which may be a bit more challenging. We tried to maintain a balance between the two types of problems, and we hope that the readers will enjoy this variation and have as much thrill and excitement as we had while creating and solving these problems.

Acknowledgements

We want to thank to Dr Ilie's students, Nicholas Jira, Vincent DeBiase, Ian Evans, and Andres Inga, who contributed to the editing (typing) of this book. We are particularly grateful to our illustrator, Julia D'Rozario, for making all of the figures. We thank Dr Ildar Sabirianov for providing useful suggestions. We thank the administration at SUNY Oswego and the office of Research and Individualized Student Experiences for overall support. We are grateful to Dr Peter Dowben, from the University of Nebraska at Lincoln, who thought that such a project has a niche. A thought of appreciation to Dr Charles Ebner, from the Ohio State University for his perfect Electrodynamics course. Also many thanks to our editors, Joel Claypool, Publisher at Morgan & Claypool Publishers, Jeanine Burke, Consulting Editor at the IOP Concise Physics e-book program, and Jacky Mucklow, Production Team Manager at the Institute of Physics. Lastly, we thank to our families and friends for their sense of humor, encouragement, and for keeping us sane and happy.

About the authors

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Carolina C Ilie is an Associate Professor with tenure at the State University of New York at Oswego. She taught Electromagnetic Theory for almost ten years and designed various problems for her students' exams, group work, and quizzes. Dr Ilie obtained her PhD in Physics and Astronomy at the University of Nebraska at Lincoln, an MSc in Physics at Ohio State University and another MSc in Physics at the University of Bucharest, Romania. She received the President's Award for Teaching Excellence in 2016 and the Provost Award for Mentoring in Scholarly and Creative Activity in 2013. She lives in Central New York with her spouse, also a physicist, and their two sons.

Photograph courtesy of James Russell/SUNY Oswego Office of Communications and Marketing.

Zachariah S Schrecengost



Zachariah S Schrecengost is a State University of New York alumnus. He graduated summa cum laude with a BS degree having completed majors in Physics, Software Engineering, and Applied Mathematics. He took the Advanced Electromagnetic Theory course with Dr Ilie and was thrilled to be involved in creating this book. He brings to the project both the fresh perspective of the student taking electrodynamics, as well as the enthusiasm and talent of an alumnus who is an electrodynamics and upper level

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Julia R D'Rozario



Julia R D'Rozario (*illustrator*) graduated from the State University of New York at Oswego in December 2016 where she completed a BS in Physics and a BA in Cinema and Screen Studies, and completed a minor in Astronomy by May 2016. She completed the Advanced Electromagnetic Theory course with Dr Ilie and has much experience of the arts through her career in film. Ms D'Rozario contributes her knowledge of electrodynamics and her talent in drawing using Inkscape software. Her future aim is to

attend graduate school and continue to combine her passions for physics and cinema.

Chapter 1

Mathematical techniques

There are a variety of mathematical techniques required to solve problems in electromagnetism. The aim of this chapter is to provide problems that will build confidence in these techniques. Concepts from vector calculus and curvilinear coordinate systems are the primary focus.

1.1 Theory

1.1.1 Dot and cross product

Given vectors $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ and $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = AB \cos \theta$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{with } |\vec{A} \times \vec{B}| = AB \sin \theta$$

where $A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$, $B = |\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$, and θ is the angle between \vec{A} and \vec{B} .

1.1.2 Separation vector

This notation is outlined by David J Griffiths in his book *Introduction to Electrodynamics* (1999, 2013). Given a source point \vec{r}' and field point \vec{r} , the separation vector points from \vec{r}' to \vec{r} and is given by

$$\vec{r} = \vec{r} - \vec{r}' = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

and the unit vector pointing from \vec{r}' to \vec{r} is

$$\hat{r} = \frac{\vec{r}}{r} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} = \frac{(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}.$$

As explained by Griffiths, this notation greatly simplifies later equations.

1.1.3 Transformation matrix

Given vector $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ in coordinate system K , the components of \vec{A} in coordinate system K' are determined by rotational matrix R given by

$$R = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}$$

with

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = R \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

1.1.4 Gradient

Given a scalar function T , the gradients for various coordinate systems are given below.

Cartesian

$$\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

Cylindrical

$$\nabla T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

Spherical

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r \sin \theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

1.1.5 Divergence

Given vector function \vec{v} , the divergences for various coordinate systems are given below.

Cartesian

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Cylindrical

$$\nabla \cdot \vec{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Spherical

$$\nabla \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

1.1.6 Curl

Given vector function \vec{v} , the curls for various coordinate systems are given below.

Cartesian

$$\nabla \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z}$$

Cylindrical

$$\nabla \times \vec{v} = \left(\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) \hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right) \hat{\phi} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{z}$$

Spherical

$$\begin{aligned} \nabla \times \vec{v} = & \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} \\ & + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi} \end{aligned}$$

1.1.7 Laplacian

Given a scalar function T , the Laplacians for various coordinate systems are given below.

Cartesian

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Cylindrical

$$\nabla^2 T = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial T}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

Spherical

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2}$$

1.1.8 Line integral

Given vector function \vec{v} and path \mathcal{P} , a line integral is given by

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot d\vec{\ell},$$

where \vec{a} and \vec{b} are the end points, and $d\vec{\ell}$ is the infinitesimal displacement vector along \mathcal{P} . In Cartesian coordinates $d\vec{\ell} = dx \hat{x} + dy \hat{y} + dz \hat{z}$.

1.1.9 Surface integral

Given vector function \vec{v} and surface \mathcal{S} , a surface integral is given by

$$\int_{\mathcal{S}} \vec{v} \cdot d\vec{a},$$

where $d\vec{a}$ is the infinitesimal area vector that has direction normal to the surface. Note that $d\vec{a}$ always depends on the surface involved.

1.1.10 Volume integral

Given scalar function T and volume \mathcal{V} , a volume integral is given by

$$\int_{\mathcal{V}} T d\tau,$$

where $d\tau$ is the infinitesimal volume element. In Cartesian coordinates $d\tau = dx dy dz$.

1.1.11 Fundamental theorem for gradients

$$\int_{\vec{a}}^{\vec{b}} (\nabla T) \cdot d\vec{\ell} = T(\vec{b}) - T(\vec{a})$$

1.1.12 Fundamental theorem for divergences (Gauss's theorem, Green's theorem, divergence theorem)

$$\int_{\mathcal{V}} (\nabla \cdot \vec{v}) d\tau = \oint_{\mathcal{S}} \vec{v} \cdot d\vec{a}$$

1.1.13 Fundamental theorem for curls (Stoke's theorem, curl theorem)

$$\int_{\mathcal{S}} (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_{\mathcal{P}} \vec{v} \cdot d\vec{\ell}$$

1.1.14 Cylindrical polar coordinates

Here our infinitesimal quantities are

$$d\vec{\ell} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}$$

and

$$d\tau = s ds d\phi dz.$$

1.1.15 Spherical polar coordinates

Here our infinitesimal quantities are

$$d\vec{\ell} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$$

and

$$d\tau = r^2 \sin \theta dr d\theta d\phi.$$

1.1.16 One-dimensional Dirac delta function

The one-dimensional Dirac delta function is given by

$$\delta(x - a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

and has the following properties

$$\int_{-\infty}^{\infty} \delta(x - a) dx = 1$$

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a)$$

$$\delta(kx) = \frac{1}{|k|} \delta(x).$$

1.1.17 Theory of vector fields

If the curl of a vector field \vec{F} vanishes everywhere, then \vec{F} can be written as the gradient of a scalar potential V :

$$\nabla \times \vec{F} \leftrightarrow \vec{F} = -\nabla V.$$

If the divergence of a vector vanishes everywhere, then \vec{F} can be expressed as the curl of a vector potential \vec{A} :

$$\nabla \cdot \vec{F} = 0 \leftrightarrow \vec{F} = \nabla \times \vec{A}.$$

1.2 Problems and solutions

Problem 1.1. Given vectors $\vec{A} = 3\hat{x} + 9\hat{y} + 5\hat{z}$ and $\vec{B} = \hat{x} - 7\hat{y} + 4\hat{z}$, calculate $\vec{A} \cdot \vec{B}$ and $\vec{A} \times \vec{B}$ using vector components and find the angle between \vec{A} and \vec{B} using both products.

Solution

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (3\hat{x} + 9\hat{y} + 5\hat{z}) \cdot (\hat{x} - 7\hat{y} + 4\hat{z}) \\ &= (3)(1) + (9)(-7) + (5)(4) = 3 - 63 + 20\end{aligned}$$

$$\vec{A} \cdot \vec{B} = -40$$

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & 9 & 5 \\ 1 & -7 & 4 \end{vmatrix} \\ &= [(9)(4) - (-7)(5)]\hat{x} + [(1)(5) - (3)(4)]\hat{y} + [(3)(-7) - (1)(9)]\hat{z} \\ \hat{A} \times \hat{B} &= 71\hat{x} - 7\hat{y} - 30\hat{z}\end{aligned}$$

To find the angle θ between \vec{A} and \vec{B} we must first calculate A and B :

$$A = \sqrt{3^2 + 9^2 + 5^2} = \sqrt{115}$$

$$B = \sqrt{1^2 + (-7)^2 + 4^2} = \sqrt{66}.$$

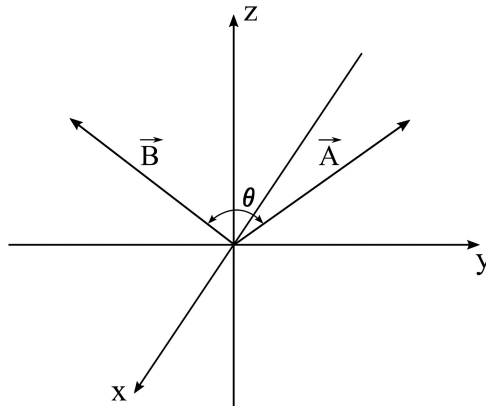
Using the dot product, we have

$$\begin{aligned}\vec{A} \cdot \vec{B} &= AB \cos \theta \rightarrow \theta = \cos^{-1} \left(\frac{-40}{\sqrt{115} \sqrt{66}} \right) \\ \theta &= 117.3^\circ.\end{aligned}$$

Using the cross product, we have

$$\begin{aligned}|\vec{A} \times \vec{B}| &= AB \sin \theta \rightarrow \sqrt{71^2 + (-7)^2 + (-30)^2} = \sqrt{115} \sqrt{66} \sin \theta \\ \theta &= 62.7^\circ.\end{aligned}$$

Note, however, that we can see that the angle between \vec{A} and \vec{B} is greater than 90° . For any argument γ , $-90^\circ \leq \sin^{-1}(\gamma) \leq 90^\circ$. Since the angle between \vec{A} and \vec{B} is greater than 90° , we must adjust for this by subtracting our angle from 180° . Therefore, $\theta = 180^\circ - 62.7^\circ = 117.3^\circ$ as expected.



Problem 1.2. The scalar triple product states $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$. Prove this by expressing each side in terms of its components.

Solution Starting with the left-hand side, the cross product is

$$\begin{aligned}\vec{B} \times \vec{C} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= (B_y C_z - B_z C_y)\hat{x} + (B_z C_x - B_x C_z)\hat{y} + (B_x C_y - B_y C_x)\hat{z}.\end{aligned}$$

Now, dotting \vec{A} with $(\vec{B} \times \vec{C})$

$$\begin{aligned}\vec{A} \cdot (\vec{B} \times \vec{C}) &= A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x) \\ &= A_x B_y C_z - A_x B_z C_y + A_y B_z C_x - A_y B_x C_z + A_z B_x C_y - A_z B_y C_x \\ &= B_x(C_y A_z - C_z A_y) + B_y(C_z A_x - C_x A_z) + B_z(C_x A_y - C_y A_x) \\ \vec{A} \cdot (\vec{B} \times \vec{C}) &= \vec{B} \cdot [(C_y A_z - C_z A_y)\hat{x} + (C_z A_x - C_x A_z)\hat{y} + (C_x A_y - C_y A_x)\hat{z}].\end{aligned}$$

Note the term in brackets is precisely $\vec{C} \times \vec{A}$, therefore

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

as desired. This procedure can easily be applied again to prove the final part of the triple product,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}).$$

Problem 1.3. Given source vector $\vec{r}' = r \cos \theta \hat{x} + r \sin \theta \hat{y}$ and field vector $\vec{r} = z\hat{z}$, find the separation vector \vec{r}'' and the unit vector \hat{r}'' .

Solution We have

$$\begin{aligned}\vec{r}'' &= \vec{r} - \vec{r}' = z\hat{z} - (r \cos \theta \hat{x} + r \sin \theta \hat{y}) \\ \vec{r}'' &= -r \cos \theta \hat{x} - r \sin \theta \hat{y} + z\hat{z}.\end{aligned}$$

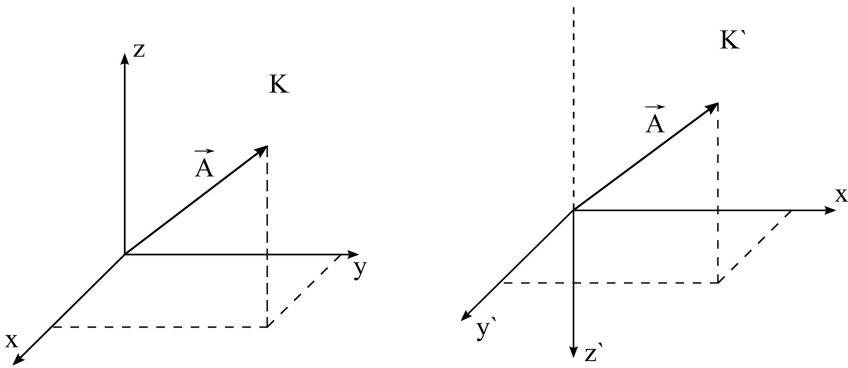
To determine the unit vector \hat{r}'' , we must first find the magnitude of \vec{r}'' ,

$$r = \sqrt{(-r \cos \theta)^2 + (-r \sin \theta)^2 + z^2} = \sqrt{r^2(\cos^2 \theta + \sin^2 \theta) + z^2} = \sqrt{r^2 + z^2}.$$

So

$$\hat{r} = \frac{\vec{r}}{r} = \frac{-r \cos \theta \hat{x} - r \sin \theta \hat{y} + z \hat{z}}{\sqrt{r^2 + z^2}}.$$

Problem 1.4. Given \vec{A} in coordinate system K , find the rotational matrix to give the components in system K' .



Solution From the figures, we have

$$A'_x = A_y, \quad A'_y = A_x, \quad A'_z = -A_z.$$

We want to find the rotational matrix R that satisfies

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = R \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

From our equations above

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

Therefore,

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Problem 1.5. Find the gradient of the following functions:

a) $T = x^4 + y^2 + z^3$

b) $T = x^2 \ln y z^3$

c) $T = x^2 y + z^3$

Solutions

a) $T = x^4 + y^2 + z^3$

$$\nabla T = \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} = 4x^3 \hat{x} + 2y \hat{y} + 3z^2 \hat{z}$$

b) $T = x^2 \ln y z^3$

$$\nabla T = \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} = 2xz^3 \ln y \hat{x} + \frac{x^2 z^3}{y} \hat{y} + 3x^2 z^2 \ln y \hat{z}$$

c) $T = x^2 y + z^3$

$$\nabla T = \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} = 2xy \hat{x} + x^2 \hat{y} + 3z^2 \hat{z}$$

Problem 1.6. Find the divergence of the following functions:

a) $\vec{v} = xy\hat{x} - 2y^2z\hat{y} + z^3\hat{z}$

b) $\vec{v} = (x + y)\hat{x} + (y + z)\hat{y} + (z + x)\hat{z}$

Solutions

a) $\vec{v} = xy\hat{x} - 2y^2z\hat{y} + z^3\hat{z}$

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = y - 4yz + 3z^2$$

b) $\vec{v} = (x + y)\hat{x} + (y + z)\hat{y} + (z + x)\hat{z}$

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 1 + 1 + 1 = 3$$

Problem 1.7. Find the curl of the following functions:

a) $\vec{v} = xy\hat{x} - 2y^2z\hat{y} + z^3\hat{z}$

b) $\vec{v} = (x + y)\hat{x} + (y + z)\hat{y} + (z + x)\hat{z}$

c) $\vec{v} = \sin x \hat{x} + \cos y \hat{y}$

Solutions

a) $\vec{v} = xy\hat{x} - 2y^2z\hat{y} + z^3\hat{z}$

$$\begin{aligned}\nabla \times \vec{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z} \\ &= \left[\frac{\partial(z^3)}{\partial y} - \frac{\partial(-2y^2z)}{\partial z} \right] \hat{x} + \left[\frac{\partial(xy)}{\partial z} - \frac{\partial(z^3)}{\partial x} \right] \hat{y} \\ &\quad + \left[\frac{\partial(-2y^2z)}{\partial x} - \frac{\partial(xy)}{\partial y} \right] \hat{z} \\ &= (0 + 2y^2)\hat{x} + (0 - 0)\hat{y} + (0 - x)\hat{z} \\ \nabla \times \vec{v} &= 2y^2\hat{x} - x\hat{z}\end{aligned}$$

b) $\vec{v} = (x + y)\hat{x} + (y + z)\hat{y} + (z + x)\hat{z}$

$$\begin{aligned}\nabla \times \vec{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z} \\ &= \left[\frac{\partial(z + x)}{\partial y} - \frac{\partial(y + z)}{\partial z} \right] \hat{x} + \left[\frac{\partial(x + y)}{\partial z} - \frac{\partial(z + x)}{\partial x} \right] \hat{y} \\ &\quad + \left[\frac{\partial(y + z)}{\partial x} - \frac{\partial(x + y)}{\partial y} \right] \hat{z} \\ \nabla \times \vec{v} &= -\hat{x} - \hat{y} - \hat{z}\end{aligned}$$

c) $\vec{v} = \sin x \hat{x} + \cos y \hat{y}$

$$\begin{aligned}\nabla \times \vec{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z} \\ &= \left[\frac{\partial(0)}{\partial y} - \frac{\partial(\cos y)}{\partial z} \right] \hat{x} + \left[\frac{\partial(\sin x)}{\partial z} - \frac{\partial(0)}{\partial x} \right] \hat{y} \\ &\quad + \left[\frac{\partial(\cos y)}{\partial x} - \frac{\partial(\sin x)}{\partial y} \right] \hat{z} = 0\end{aligned}$$

Problem 1.8. Prove $\nabla \times (\nabla T) = 0$.

Solution

$$\begin{aligned} \nabla \times (\nabla T) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y} \left(\frac{\partial T}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial T}{\partial y} \right) \right] \hat{x} + \left[\frac{\partial}{\partial z} \left(\frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial z} \right) \right] \hat{y} \\ &\quad + \left[\frac{\partial}{\partial x} \left(\frac{\partial T}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial x} \right) \right] \hat{z} \\ \nabla \times (\nabla T) &= 0. \end{aligned}$$

Problem 1.9. Find the Laplacian of the following functions:

- a) $T = x + y^2 + xz + 3$
- b) $T = e^x + \sin y \cos(2z)$
- c) $T = \sin x \cos y$
- d) $\vec{v} = xy\hat{x} + z^2\hat{y} - 2\hat{z}$

Solutions

a) $T = x + y^2 + xz + 3$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 + 2 + 0 = 2$$

b) $T = e^x + \sin y \cos(2z)$

$$\begin{aligned} \nabla^2 T &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = e^x - \sin y \cos(2z) - 4 \sin y \cos(2z) \\ &= e^x - 5 \sin y \cos(2z) \end{aligned}$$

c) $T = \sin x \cos y$

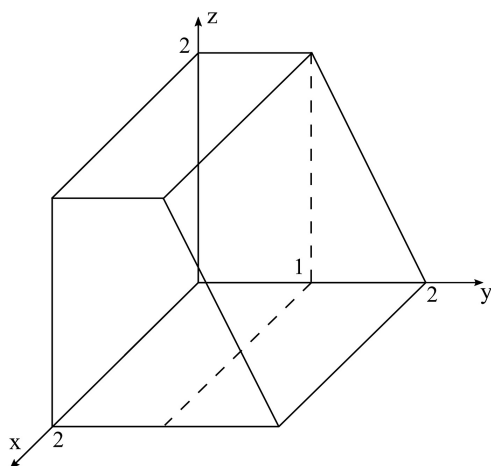
$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = -\sin x \cos y - \sin x \cos y = -2 \sin x \cos y$$

$$d) \vec{v} = xy\hat{x} + z^2\hat{y} - 2z\hat{z}$$

$$\begin{aligned} \nabla^2 \vec{v} &= \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) \hat{x} + \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) \hat{y} \\ &\quad + \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \hat{z} \end{aligned}$$

$$\nabla^2 \vec{v} = (0 + 0 + 0)\hat{x} + (0 + 0 + 2)\hat{y} + (0 + 0 + 0)\hat{z} = 2\hat{y}$$

Problem 1.10. Test the divergence theorem with $\vec{v} = 2xy\hat{x} + y^2z^3\hat{y} + (x^2z - 2y)\hat{z}$ and the volume below.



Solution The divergence theorem states

$$\int_V \nabla \cdot \vec{v} d\tau = \oint_S \vec{v} \cdot d\vec{a}.$$

Starting with the left-hand side, we have the divergence

$$\nabla \cdot \vec{v} = 2y + 2yz^3 + x^2 = 2y(z^3 + 1) + x^2.$$

We must split the volume into two pieces, (a) $0 \leq y \leq 1$ and (b) $1 \leq y \leq 2$.

(a)

$$\int \nabla \cdot \vec{v} d\tau = \int_0^2 \int_0^1 \int_0^1 [2y(z^3 + 1) + x^2] dy dx dz = \frac{52}{3}$$